

Two-parameter extensions of the κ -Poincaré quantum deformation

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Abstract

We consider the extensions of classical r -matrix for κ -deformed Poincaré algebra which satisfy modified Yang-Baxter equation. Two examples introducing additional deformation parameter (dimensionfull $\frac{1}{\kappa}$ or dimensionless ξ) are presented. We describe the corresponding quantization (two-parameter κ -Poincaré quantum Hopf algebras) in explicit form as obtained by twisting of standard κ -deformed framework. In the second example quantum twist function depends on nonclassical generators, with κ -deformed coproduct. Finally we mention also the “soft” twists with carrier in fourmomenta sector.

1 Introduction

Recently many authors (see e.g. [1]–[13]) considered the deformations of relativistic symmetries in the Hopf-algebraic framework of quantum groups [14]–[16]. In particular physically appealing are the κ -deformations [1], [3, 4], [8, 9], [13] which introduce mass-like deformation parameter κ usually linked with Planck mass M_P (e.g. $\kappa = M_P$) as well as with the quantum gravity corrections.¹ It appears interesting to look for multiparameter extensions of standard κ -deformed framework and present the result as multiparameter Hopf-algebraic structure.

Quantum groups infinitesimally are characterized by the classical r -matrices. In particular standard κ -deformation of Poincaré algebra [1], [3, 4], [8, 9] is described by the following classical r -matrix ($i = 1, 2, 3$)

$$r = \frac{1}{\kappa} N_i \wedge P_i, \quad (1.1)$$

where $M_{\mu\nu} = (M_i = \frac{1}{2}\epsilon_{ijk} M_{jk}, N_i = M_{i0})$ denote the Lorentz algebra generators and $P_\mu = (P_i, P_0)$ represent the Abelian fourmomenta. More explicitly

$$[M_i, M_j] = i\epsilon_{ijk} M_k, \quad [N_i, N_j] = -i\epsilon_{ijk} M_k, \quad (1.2a)$$

$$[M_i, N_j] = i\epsilon_{ijk} N_k,$$

$$[M_i, P_j] = i\epsilon_{ijk} P_k, \quad [M_i, P_0] = 0, \quad (1.2b)$$

$$[N_i, P_j] = i\delta_{ij} P_0, \quad [N_i, P_0] = iP_i.$$

The classical r -matrix (1.1) satisfies the following modified Yang-Baxter equation (MYBE)

$$[[r, r]] \equiv [r_{12}, r_{13}] + \dots = \frac{1}{\kappa^2} M_{\mu\nu} \wedge P^\mu \wedge P^\nu, \quad (1.3)$$

where $r_{12} = \frac{1}{\kappa} N_i \wedge P_i \wedge 1$, $r_{13} = \frac{1}{\kappa} N_i \wedge 1 \wedge P_i$ and $r_{23} = \frac{1}{\kappa} 1 \wedge N_i \wedge P_i$. We found that the following ansatz provides new solutions of (1.3) (ξ_1 and ξ_2 have the inverse mass dimensions)

$$\tilde{r} = r + \delta r \quad \delta r = \xi_1 M_3 \wedge P_0 + \xi_2 M_3 \wedge P_+, \quad (1.4)$$

¹The kinematic consequences of such deformation has been studied in the framework of so-called doubly special relativity (DSR) theories (see. e.g. [17]–[19]).

where $P_{\pm} = P_1 \pm iP_2$. The relation

$$[[r, \delta r]] + [[\delta r, r]] + [[\delta r, \delta r]] = 0, \quad (1.5)$$

is valid if

$$\xi_2(\xi_1 - \frac{1}{\kappa}) = 0, \quad (1.6)$$

and the classical r -matrices satisfying again the MYBE given by (1.3) are the following

$$\begin{aligned} \text{i) } \quad \xi_2 = 0, \quad \xi_1 = \frac{1}{\tilde{\kappa}} \\ \delta r_1 = \frac{1}{\tilde{\kappa}} M_3 \wedge P_0, \end{aligned} \quad (1.7a)$$

This modification of κ -Poincaré bi-algebra has been also mentioned in [20].

$$\begin{aligned} \text{ii) } \quad \xi_2 = \xi \frac{1}{\kappa}, \quad \xi_1 = \frac{1}{\kappa} \quad (\xi \text{ is dimensionless}) \\ \delta r_2 = \frac{1}{\kappa} (M_3 \wedge P_0 + \xi M_3 \wedge P_+). \end{aligned} \quad (1.7b)$$

The aim of our paper is to describe explicitly the quantization of the classical r -matrices described by (1.4) and (1.7a–1.7b) by providing corresponding Hopf algebra formulae.

We shall introduce the Drinfeld twist function or twisting element \mathcal{F} which satisfies the κ -deformed twist equations [21]

$$\mathcal{F}_{12}(\Delta_{\kappa} \otimes \text{id})\mathcal{F} = \mathcal{F}_{23}(\text{id} \otimes \Delta_{\kappa})\mathcal{F}, \quad (1.8a)$$

with the following linear term in the series expansion in deformation parameters ($\delta r = \delta r_k^{(1)} \wedge \delta r_k^{(2)}$)

$$\mathcal{F} = 1 \otimes 1 + \delta r_k^{(1)} \otimes \delta r_k^{(2)} + \dots \quad (1.8b)$$

Twisting element $\mathcal{F} \in U_{\kappa}(\mathcal{P}^4) \otimes U_{\kappa}(\mathcal{P}^4)$ maps the κ -deformed Poincaré-Hopf algebra $\mathcal{A}_{\kappa}(m, \Delta_{\kappa}, S_{\kappa}, \eta, \varepsilon)$ into two-parameter κ -deformed Poincaré-Hopf algebra $\mathcal{A}_{\kappa, \alpha}(m, \Delta_{\kappa, \alpha}, S_{\kappa, \alpha}, \eta, \varepsilon)$ where $\alpha = \frac{1}{\kappa}$ or $\alpha = \xi$

$$\Delta_{\kappa, \alpha} = \mathcal{F} \Delta_{\kappa}(a) \mathcal{F}^{-1}, \quad (1.9)$$

$$S_{\kappa,\alpha}(a) = u S_{\kappa}(a) u^{-1}, \quad (1.10)$$

and $(\mathcal{F} \equiv \sum_i \mathcal{F}_i^{(1)} \otimes \mathcal{F}_i^{(2)})$.

$$u = \sum_i \mathcal{F}_i^{(1)} \cdot S(\mathcal{F}_i^{(2)}). \quad (1.11)$$

It appears that our twist function \mathcal{F} satisfies more specific factorized twist equations [22]

$$(\Delta_{\kappa} \otimes \text{id}) \mathcal{F} = \mathcal{F}_{13} \cdot \mathcal{F}_{23}, \quad (1.12a)$$

$$(\text{id} \otimes \Delta_{\kappa,\alpha}) \mathcal{F} = \mathcal{F}_{12} \cdot \mathcal{F}_{13}, \quad (1.12b)$$

where in eq. (1.12b) one should insert the twisted coproduct (1.9). Indeed, one can show for the general case of deformed coproduct (1.9) that the relation (1.8a) follows from (1.12a-1.12b).

The plan of our paper is the following:

In Sect. 2 we describe the standard κ -deformation using the bicrossproduct basis and find the solutions to the twist equations corresponding to the terms δr_1 (see 1.7a) and δr_2 (see 1.7b). In Sect. 3 the two-parameter κ -deformed Poincaré algebra is constructed in the explicit form. In Sect. 4 we comment on the possibility of inserting “soft” twist, with carrier algebra described by a fourmomentum sector.

We would like to stress here that usually the twists are classical, i.e. the carrier algebra of the twist function is a classical Lie algebra, with primitive coproduct. Recently however, there were also considered twists of quantum algebras, e.g. quantum Jordanian twist of the q -deformed $sl(2)$ Borel subalgebra [23]; see also [24]–[25]. In this paper by quantizing classical r -matrix (1.7b) we provide another example of quantum (i.e. nonclassical) twist.

2 κ -deformed Poincaré algebra in bicrossproduct basis and the twist function \mathcal{F}

The κ -deformed Poincaré Hopf algebra $A_{\kappa}(m, \Delta_{\kappa}, S_{\kappa}, \eta, \varepsilon)$ can be written in different bases e.g. the standard one [4], bicrossproduct basis [9] and the classical Poincaré algebra basis [26]. Usually the twist is performed in classical Lie algebra basis; here we propose to consider the twist in nonclassical bicrossproduct basis.

The κ -deformed bicrossproduct basis is described in algebraic sector by
i) classical Lorentz algebra (see (1.2a))
ii) the cross relations of Lorentz generators and fourmomenta (1.2b) with only one deformed relation

$$[N_i, P_j] = i \delta_{ij} \left[\frac{\kappa}{2} \left(1 - e^{-\frac{2P_0}{\kappa}} \right) + \frac{1}{2\kappa} \vec{P} \right] - \frac{i}{\kappa} P_i P_j. \quad (2.1)$$

The coalgebra sector is described by the following coproducts:

$$\begin{aligned} \Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\ \Delta(P_i) &= P_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_i, \\ \Delta(M_i) &= M_i \otimes 1 + 1 \otimes M_i, \\ \Delta(N_i) &= N_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes N_i - \frac{1}{\kappa} \varepsilon_{ijk} M_j \otimes P_k. \end{aligned} \quad (2.2)$$

The solution of the twist equations (1.8a–1.8b) has the following form:

a) For the classical r -matrix (1.7a)

$$\mathcal{F}_1(\tilde{\kappa}) = e^{\frac{1}{\kappa} M_3 \otimes P_0}. \quad (2.3)$$

Using (1.11) one gets

$$u(\tilde{\kappa}) = e^{-\frac{1}{\kappa} M_3 P_0} \quad (2.4)$$

b) For the classical r -matrix (1.7b)

$$\begin{aligned} \mathcal{F}_2(\kappa, \xi) &= e^{\frac{1}{\kappa} M_3 \otimes P_0} e^{M_3 \otimes \ln(1 + \xi P_+)} \\ &= e^{M_3 \otimes \ln\left(e^{\frac{P_0}{\kappa}} + \xi P_+ e^{\frac{P_0}{\kappa}}\right)}, \end{aligned} \quad (2.5)$$

and further

$$u(\kappa, \alpha) = e^{M_3 \ln\left(e^{-\frac{1}{\kappa} P_0} - \xi P_+\right)}. \quad (2.6)$$

The twist (2.3) satisfies (1.12a–b) as follows from the general property of commuting primitive generators [22]. The solution (2.5) is less trivial. To check its validity consider the 3-dimensional Hopf algebra $\mathcal{A} = (C, D, H)$

$$[H, D] = D, \quad [C, H] = [C, D] = 0, \quad (2.7)$$

with nonprimitive coproducts

$$\begin{aligned}\Delta(H) &= H \otimes 1 + 1 \otimes H, & \Delta(C) &= C \otimes C, \\ \Delta(D) &= D \otimes 1 + C \otimes D.\end{aligned}\tag{2.8}$$

We assert that the function

$$\mathcal{F} = e^{H \otimes \ln(C+D)}.\tag{2.9}$$

is the twisting element for \mathcal{A} and satisfies the factorised equations (1.12a–b). In fact the first of these equations is satisfied due to primitivity of H . To verify the second equation it is sufficient to prove that the twisted coproduct $\mathcal{F}(\Delta(C+D))\mathcal{F}^{-1}$ is group-like. Such a statement is the consequence of the relations (2.7–2.8).

In our case the Hopf algebra $\mathcal{A} = (\{C, D, H\}, \Delta = \Delta_\kappa)$ is represented by

$$C = e^{\frac{P_0}{\kappa}}, \quad D = P_+ e^{\frac{P_0}{\kappa}}, \quad H = M_3.\tag{2.10}$$

Thus we have demonstrated that \mathcal{F}_2 is the twisting element for the κ -deformed Poincaré algebra.

3 Two-parameter κ -deformed Poincaré algebra

i) Reshetikhin twist (2.3) (two mass-like deformation parameters κ and $\tilde{\kappa}$):

In such a case the modified coproducts $\Delta_{\kappa, \tilde{\kappa}}$ look as follows (we introduce $M_\pm = M_1 \pm i M_2$, $N_\pm = N_1 \pm i N_2$ and $P_\pm = P_1 \pm i P_2$)

$$\begin{aligned}
\Delta_{\kappa,\tilde{\kappa}}(M_{\pm}) &= M_{\pm} \otimes e^{\pm \frac{1}{\kappa} P_0} + 1 \otimes M_{\pm}, \\
\Delta_{\kappa,\tilde{\kappa}}(M_3) &= M_3 \otimes 1 + 1 \otimes M_3, \\
\Delta_{\kappa,\tilde{\kappa}}(N_{\pm}) &= N_{\pm} \otimes e^{\left(\pm \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}}\right) P_0} + 1 \otimes N_{\pm} + i \left(\mp \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}}\right) M_3 \otimes P_{\pm} \\
&\quad \pm i \frac{1}{\kappa} M_{\pm} \otimes P_3 e^{\pm \frac{1}{\kappa} P_0}, \\
\Delta_{\kappa,\tilde{\kappa}}(N_3) &= N_3 \otimes e^{-\frac{1}{\kappa} P_0} + 1 \otimes N_3 - i \frac{1}{\tilde{\kappa}} M_3 \otimes P_3 \\
&\quad - i \frac{1}{2\kappa} M_+ \otimes P_- e^{\frac{1}{\kappa} P_0} + i \frac{1}{2\kappa} M_- \otimes P_+ e^{-\frac{1}{\kappa} P_0}, \\
\Delta_{\kappa,\tilde{\kappa}}(P_{\pm}) &= P_{\pm} \otimes e^{\left(\pm \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}}\right) P_0} + 1 \otimes P_{\pm}, \\
\Delta_{\kappa,\tilde{\kappa}}(P_3) &= P_3 \otimes e^{-\frac{1}{\kappa} P_0} + 1 \otimes P_3, \\
\Delta_{\kappa,\tilde{\kappa}}(P_0) &= P_0 \otimes 1 + 1 \otimes P_0.
\end{aligned} \tag{3.1}$$

Using (2.4) one gets the following formulae for antypodes:

$$\begin{aligned}
S_{\kappa,\tilde{\kappa}}(M_{\pm}) &= -M_{\pm} e^{\mp \frac{1}{\kappa} P_0}, & S_{\kappa,\tilde{\kappa}}(M_3) &= -M_3, \\
S_{\kappa,\tilde{\kappa}}(N_{\pm}) &= -\left(N_{\pm} - i \left(\mp \frac{1}{\kappa} - \frac{1}{\tilde{\kappa}}\right) M_3 P_{\pm} \mp i \frac{1}{\kappa} M_{\pm} P_3\right) e^{\left(\mp \frac{1}{\kappa} + \frac{1}{\tilde{\kappa}}\right) P_0}, \\
S_{\kappa,\tilde{\kappa}}(N_3) &= -\left(N_3 + i \frac{1}{\kappa} M_3 P_3 + i \frac{1}{2\kappa} (M_+ P_- - M_- P_+)\right) e^{\frac{1}{\kappa} P_0}, \\
S_{\kappa,\tilde{\kappa}}(P_{\pm}) &= -P_{\pm} e^{\left(\mp \frac{1}{\kappa} + \frac{1}{\tilde{\kappa}}\right) P_0}, \\
S_{\kappa,\tilde{\kappa}}(P_3) &= -P_3 e^{\frac{1}{\kappa} P_0}, & S_{\kappa,\tilde{\kappa}}(P_0) &= -P_0.
\end{aligned} \tag{3.2}$$

Only two generators (P_0, M_3) have primitive coproducts. It should be noticed that if $\kappa = \tilde{\kappa}$ ($\kappa = -\tilde{\kappa}$) then additionally the generator P_+ (P_-) becomes primitive.

ii) Complex twist (2.5) (one mass-like deformation parameter κ and one dimensionless ξ).

One gets the following coproducts ($\omega(\xi) = \ln(1 + \xi P_+)$)

$$\begin{aligned}
\Delta_{\kappa,\xi}(M_+) &= M_+ \otimes e^{\frac{1}{\kappa}P_0+\omega(\xi)} + 1 \otimes M_+, \\
\Delta_{\kappa,\xi}(M_-) &= M_- \otimes e^{-\frac{1}{\kappa}P_0-\omega(\xi)} + 1 \otimes M_- + 2\xi M_3 \otimes P_3 e^{-\omega(\xi)}, \\
\Delta_{\kappa,\xi}(M_3) &= M_3 \otimes e^{-\omega(\xi)} + 1 \otimes M_3, \\
\Delta_{\kappa,\xi}(N_+) &= N_+ \otimes e^{\omega(\xi)} + 1 \otimes N_+ - \frac{i}{\kappa} M_3 \otimes P_+ \left(1 + e^{\omega(\xi)}\right) \\
&\quad + \frac{i}{\kappa} M_+ \otimes P_3 e^{\frac{P_0}{\kappa}+\omega(\xi)}, \\
\Delta_{\kappa,\xi}(N_-) &= N_- \otimes e^{-\frac{2P_0}{\kappa}-\omega(\xi)} + 1 \otimes N_- - i\xi M_3 \otimes \left(2P_0 e^{-\omega(\xi)} + \frac{P_-}{\kappa} P_+\right) \\
&\quad - \frac{i}{\kappa} M_- \otimes P_3 e^{-\frac{P_0}{\kappa}-\omega(\xi)}, \\
\Delta_{\kappa,\xi}(N_3) &= N_3 \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes N_3 - \frac{i}{\kappa} M_3 \otimes P_3 e^{-\omega(\xi)} \\
&\quad + \frac{i}{2\kappa} \left(M_- \otimes P_+ e^{-\frac{P_0}{\kappa}-\omega(\xi)} - M_+ \otimes P_- e^{\frac{P_0}{\kappa}+\omega(\xi)} \right), \\
\Delta_{\kappa,\xi}(P_+) &= P_+ \otimes e^{\omega(\xi)} + 1 \otimes P_+, \\
\Delta_{\kappa,\xi}(P_-) &= P_- \otimes e^{-\frac{2P_0}{\kappa}-\omega} + 1 \otimes P_-, \\
\Delta_{\kappa,\xi}(P_3) &= P_3 \otimes e^{-\frac{2P_0}{\kappa}} + 1 \otimes P_3, \\
\Delta_{\kappa,\xi}(P_0) &= P_0 \otimes 1 + 1 \otimes P_0.
\end{aligned} \tag{3.3}$$

Using (2.6) and (1.10) one can calculate also the antipodes.

We see that only one coproduct of the Poincaré generators, for P_0 , remains primitive. Second primitive product is provided by the function $\omega(\xi)$ belonging to the enveloping algebra of \mathcal{P}_4 .

It should be added that the twist (2.5) is not unitary, i.e. it extends the real coproducts (2.2) in a way which does not preserve the reality of the Poincaré generators.

4 Final Remarks

The aim of this paper was to provide two examples of twisting of standard κ -Poincaré algebra. In accordance with twisting quantization scheme only the coproducts and atipodes are modified. The second twist is less attractive

for physical applications (the coproducts become complex), but provides an interesting example of a twist with deformed carrier algebra.

In this note we neglected the Abelian twists with carriers in the translation generators sector. It is possible to perform such deformations in the initial κ -Poincaré algebra by adding to (1.1) the following classical r -matrix (ξ_i - constant three-vector):

$$\delta \tilde{r} = \frac{1}{\kappa} \xi_i P_0 \wedge P_i, \quad (4.1)$$

which can be achieved by twisting the κ -Poincaré algebra by the following twist factor

$$\mathcal{F}_R(\zeta_i) = e^{\zeta_i P_i \otimes \left(e^{-\frac{1}{\kappa} P_0} - 1\right)}. \quad (4.2)$$

In the light of recent results [27, 28] which link such type of twist with Seiberg-Witten θ -deformation corresponding to constant value of $\theta_{\mu\nu}$ in the commutator of space-time coordinates ($[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$), such “soft” quantum deformation of κ -Poincaré algebra might be also significant.

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